SNAP 2017. Laplace's equation and conformal maps.

Problem Set 2

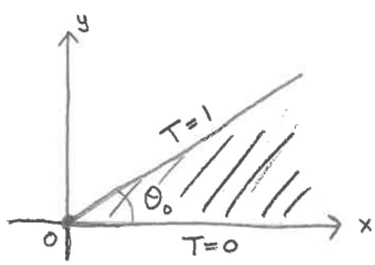
1. Let $H(u,v)$ be a harmonic function (i.e. $H_{uu}+H_{vv}=0$). Let $f(z)=u(x,y)+iv(x,y)$ be a holomorphic function, where we are writing $z=x+iy$ as usual. Show directly using the Chain rule and the Cauchy-Riemann equations that $h(x,y)=H(u(x,y),v(x,y))$ is a harmonic function of x and y where defined.

   Solution. $h_x=H_u u_x+H_v v_x$ and $h_{xx}=H_{uu}u_x^2+2H_{uv}u_x v_x+H_{vv}v_x^2+H_u u_{xx}+H_v v_{xx}$. Similarly, $h_{yy}=H_{uu}u_y^2+2H_{uv}u_y v_y+H_{vv}v_y^2+H_u u_{yy}+H_v v_{yy}$. Using the Cauchy-Riemann equations $u_x=v_y$ and $v_x=-u_y$ we get $h_{yy}=H_{uu}v_x^2-2H_{uv}v_x u_x+H_{vv}u_x^2+H_u u_{yy}+H_v v_{yy}$.

   $$h_{xx}+h_{yy}=(u_x^2+v_x^2)(H_{uu}+H_{vv})+H_u(u_{xx}+u_{yy})+H_v(v_{xx}+v_{yy})=0,$$

   recalling that u,v and H are harmonic.

2. Fix an angle θ_0 with $0<\theta_0<\pi/2$ and consider the wedge shaped region as shown. Find the steady state heat distribution $T(x,y)$ if the top edge is held at temperature $T=1$ and the bottom edge is held at temperature $T=0$.

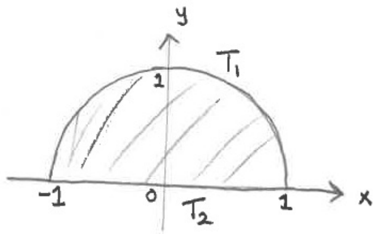
   

   Solution. $T=\frac{1}{\theta_0}\arctan(y/x)$.

3. Find the steady state heat distribution $T(x,y)$ on a half disk $\{x^2+y^2<1,\ y>0\}$ if the top semi-circular edge is held at a temperature T_1 and the bottom edge $\{(x,0)\mid -1<x<1\}$ is held at a temperature T_2.

   Show that the isotherms (curves with $T=$ constant) are arcs of circles with center on the y-axis, and which pass through the points $(-1,0)$ and $(1,0)$.

   Hint: consider the full disk with the lower semi-circle held at temperature T_3 with $T_2=\frac{1}{2}(T_1+T_3)$.



Solution. As in the hint we consider the unit disk with top edge at temperature T_1 and bottom edge $T_3 = 2T_2 - T_1$. Map to the upper half plane via $z \mapsto -i(z-1)/(z+1)$ and then map to a horizontal strip via $z \mapsto \operatorname{Log} z$. We get a linear solution on the horizontal strip, giving

$$T(z) = T_1 + \frac{2(T_2 - T_1)}{\pi} \operatorname{Arg}\left(\frac{i(1-z)}{1+z}\right).$$

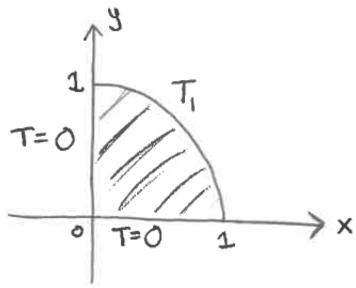
For the isotherms, write

$$\frac{i(1-z)}{z+1} = \frac{1}{(x+1)^2 + y^2}\left(2y - i(x^2 + y^2 - 1)\right),$$

so the argument is constant if $x^2 + y^2 - 1 = 2yc$ for a constant c, which is the equation of a circle centered at $(0, c)$, which contains $(1, 0)$ and $(-1, 0)$, namely

$$x^2 + (y - c)^2 = c^2 + 1.$$

4. Find the steady state heat distribution $T(x, y)$ on the quarter disk $\{x^2 + y^2 < 1,\ y > 0,\ x > 0\}$ if the quarter-circular edge is held at a temperature T_1 and the two straight edges along the x and y-axes are held at a temperature zero.

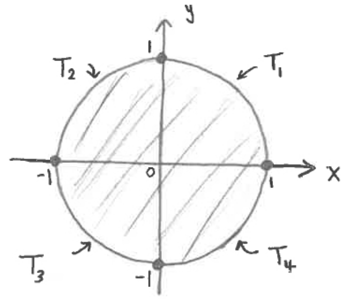


Solution. Apply $z \mapsto z^2$ and use the previous problem to get

$$T(z) = T_1 - \frac{2T_1}{\pi} \operatorname{Arg} \frac{i(1-z^2)}{1+z^2}.$$

5. Find the steady state heat distribution $T(x,y)$ on the unit disk $\{x^2+y^2<1\}$ if the quarter circles are held at constant temperatures T_1, T_2, T_3, T_4 as shown.

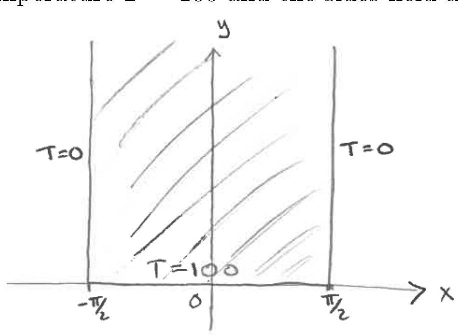
   Hint: map to the upper half plane and consider solutions of the form $w \mapsto Arg(w-a)$.



   Solution. Map to the upper half plane via $w = i(1-z)/(1+z)$ and use the hint with $a = -1, 0, 1$ to get

$$T = \frac{(T_1-T_2)}{\pi}\mathrm{Arg}\left(\frac{i(1-z)}{1+z}+1\right) + \frac{(T_4-T_1)}{\pi}\mathrm{Arg}\left(\frac{i(1-z)}{1+z}\right)$$
$$+\frac{(T_3-T_4)}{\pi}\mathrm{Arg}\left(\frac{i(1-z)}{1+z}-1\right)+T_2.$$

6. Find the steady state heat distibution $T(x,y)$ on the semi-infinite vertical slab $\{(x,y) \mid -\pi/2 < x < \pi/2,\ y > 0\}$, with the bottom edge held at temperature $T = 100$ and the sides held at temperature zero.



   Solution. Map by $w = \sin z$ to the upper half plane and then consider $w \mapsto \mathrm{Arg}(w-a)$ for $a = \pm 1$ to get

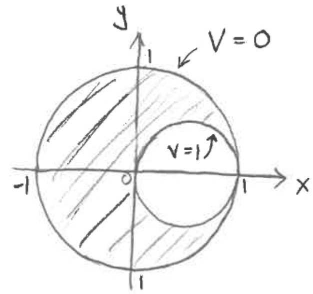
$$T = \frac{100}{\pi}\left(\mathrm{Arg}(\sin z - 1) - \mathrm{Arg}(\sin z + 1)\right).$$

7. Let $V(x,y)$ be the potential function for the electric field for a conducting laminar plate corresponding to the lunar domain

$$\{x^2+y^2<1\} \cap \{(x-1/2)^2+y^2>1/4\}$$

with boundary values $V = 0$ on the unit circle and $V = 1$ on the smaller circle $(x - 1/2)^2 + y^2 = 1/4$. Solve for $V(x, y)$ and show that the equipotential curve $V(x, y) = c$ is a circle centered at $(c/(1 + c), 0)$ with radius $1/(1 + c)$.

Hint: map the domain to a horizontal strip with a Möbius transformation that sends 1 *to* ∞



Solution. Apply $w = (1-i)\frac{z-i}{z-1}$ which maps the domain to the horizontal strip $0 < v < 1$ (where $w = u + iv$) with boundary values $V = 1$ on $v = 1$ and $V = 0$ on $v = 0$, and which has solution $V = v$. Hence the solution in the original domain is $V(x, y) = \mathrm{Im}\left((1 - i)\frac{z-i}{z-1}\right) = \frac{1-x^2-y^2}{(x-1)^2+y^2}$, and $V = c$ can be rewritten

$$\left(x - \frac{c}{1+c}\right)^2 + y^2 = \frac{1}{(1+c)^2}.$$